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LETTER TO THE EDITOR

One kind of the perturbed Korteweg–de Vries equation

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Abstract. Analytical properties of a new kind of perturbed Korteweg–de Vries equation with a term of the Hilbert transformation with respect to an unknown function are investigated. It is shown that such an equation possesses periodic and solitary wave solutions and those solutions are modulationally unstable.

The unidirectional propagation of a long surface wave in a finite liquid layer on an undeformable porous bottom saturated by the same liquid, is described by a specific kind of perturbed Korteweg–de Vries (PKdV) equation (Korsunsky 1991, 1993). Namely, evolution of the quantity $\eta(x, t)$, which is a derivation of the free liquid surface, is determined by the equation

$$\eta_t + (1 + 3\alpha\eta/2)\eta_x + \beta\eta_{xxx}/6 = -\varepsilon H[\eta]/2 \tag{1}$$

where $\alpha = a/\lambda$ is the nonlinearity parameter, $\beta = h^2/\lambda^2$ is the dispersion parameter, ε characterizes the number of pores in a unit volume of the porous media $(0 < \varepsilon < 1)$, a is the wave amplitude, λ is the wavelength, h is the depth of the liquid layer, $H[\eta]$ is the Hilbert transformation

$$H[\eta] = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\eta(x', t)}{x' - x}\, \mathrm{d}x'$$

and $\fint_{-\infty}^{\infty}$ denotes the Cauchy principal value integral. Another well-known equation which includes the Hilbert transformation is the so-called Benjamin–Ono equation (Ablowitz and Segur 1981), a specific class of nonlinear singular integral-differential equations which was investigated by Ablowitz *et al.* (1987).

In this letter we discuss some analytical properties of the perturbed KdV equation (1), hereafter referred to as the H-perturbed KdV equation. We find the periodic solution of this equation and investigate its modulational stability.

First of all we note that the term on the right-hand side of equation (1) does not affect the conservancy of wave mass and energy

$$\partial/\partial t \int_{-\infty}^{\infty} \eta\, \mathrm{d}x = 0 \qquad \partial/\partial t \int_{-\infty}^{\infty} \eta^2\, \mathrm{d}x = 0$$

if $\eta, \eta_x \to 0$, $|x| \to \infty$.

Consider the H-perturbed KdV-equation in the form

$$u_t + u_x + \alpha u u_x + \beta u_{xxx}/6 = -\varepsilon H[u]/2 \tag{2}$$

0305-4470/93/170859+03 $07.50

where $u=3\eta/2$. Let us look for a travelling-wave solution to this equation of the form $u(x,t)=u(\vartheta)$, $\vartheta=kx-\omega t$. We obtain

$$(\omega-k)u'-\alpha kuu'-\beta k^3u'''/6=\varepsilon H[u]/2 \tag{3}$$

where we put

$$\begin{aligned} u(\vartheta;\alpha)&=u_1(\vartheta)+\alpha u_2(\vartheta)+\alpha^2u_2(\vartheta)+\cdots \\ \omega(k;\alpha)&=\omega_0(k)+\alpha\omega_1(k)+\alpha^2\omega_2(k)+\cdots. \end{aligned} \tag{4}$$

Using (4), equation (3) gives the hierarchy

$$(\omega_0-k)u_1'-\beta u_1'''/6-\varepsilon H[u_1]/2=0 \tag{5}$$

$$(\omega_0-k)u_2'-\beta u_2'''/6-\varepsilon H[u_2]/2=ku_1u_1'-\omega_1u_1' \tag{6}$$

$$(\omega_0-k)u_3'-\beta u_3'''/6-\varepsilon H[u_3]/2=k(u_1u_2)'-\omega_1u_2'-\omega_2u_1'. \tag{7}$$

We find from equation (5) the linear part of the solution

$$u_1=\alpha\cos\vartheta \qquad \omega_0=k-\beta k^3/6+\varepsilon/2. \tag{8}$$

Using (8), the removal of the secular terms in the right-hand side of equation (6) requires $\omega_1=0$ and then the solution of equation (6) is given by

$$u_2=\frac{\alpha^2k}{\varepsilon+2\beta k^3}\cos 2\vartheta. \tag{9}$$

Using (8) and (9) in the next step, the condition of removal of secular terms in equation (7) then gives

$$u_3=\frac{3\alpha^3k^2}{2(\varepsilon+2\beta k^3)(\varepsilon+4\beta k^3)}\cos 3\vartheta \tag{10}$$

$$\omega_2=\frac{\alpha^2k^2}{2(\varepsilon+2\beta k^3)}. \tag{11}$$

Thus

$$u=\alpha\cos\vartheta+\alpha\frac{ak^2}{\varepsilon+2\beta k^3}\cos 2\vartheta+\alpha^2\frac{3\alpha^2k^2}{2(\varepsilon+2\beta k^3)(\varepsilon+4\beta k^3)}\cos 3\vartheta+O(\alpha^3) \tag{12}$$

$$\omega=k-\frac{\beta k^3}{6}+\frac{\varepsilon}{2}+\alpha^2\frac{\alpha^2k^2}{2(\varepsilon+2\beta k^3)}+O(\alpha^3). \tag{13}$$

Relations (12) and (13) shows that the H-perturbed KdV equation possesses purely periodic wave solutions with the nonlinear effects showing up as an amplitude-dependent frequency shift and higher harmonics generation.

Following the Whitham results (Whitham 1974) we say that if the dispersion relation for a nonlinear dispersive wave of amplitude α is of the form

$$\omega=\omega_0(k)+\alpha^2\omega_2(k) \tag{14}$$

and the dispersion is negative (i.e. $\mathrm{d}^2\omega_0/\mathrm{d}k^2<0$) then the wave train is modulationally unstable if

$$\omega_2>0. \tag{15}$$

Both conditions are valid for solution (12) and (13), so, this wave train is modulationally unstable.

This result suggests that in such a system soliton-type stationary waves can exist. In order to investigate this type of wave we rewrite equation (1) in the form

$$\varphi_t + \varphi\varphi_\xi + \beta\varphi_{\xi\xi\xi}/6 = -\varepsilon H[\varphi]/2 \tag{16}$$

where $\varphi = 3\alpha\eta/2$, $\varphi = \varphi(\xi, t)$, $\xi = x - t$, and we introduce a small parameter $\delta \ll 1$ and new independent variables

$$\xi = \delta(\xi - ct) \qquad \tau = \delta^2 t \qquad \vartheta = k\xi - \omega t.$$

We look for solutions of the form (Shivamoggi 1989)

$$\varphi = \sum_{n=1}^{\infty} \delta^n \sum_{m=-\infty}^{\infty} \varphi_{nm}(\xi, \tau)\exp(im\vartheta). \tag{17}$$

Inserting (17) in (16) yields to order $O(\delta)$

$$m = \pm 1: \omega = \varepsilon/2 - \beta k^3/6 \tag{18}$$

$$m \neq \pm 1: \varphi_{1m} = 0. \tag{19}$$

To order $O(\delta^2)$, using (18) and (19), we obtain

$$m = 0: \varphi_{20} = 0 \tag{20}$$

$$m = \pm 1: c = -\beta k^2/2 \tag{21}$$

$$m = 2: \varphi_{22} = k|\varphi_{11}|^2/2(\varepsilon + 2\beta k^2). \tag{22}$$

In the next step for (18)–(22) we find to order $O(\varepsilon^3)$

$$m = 1: \mathrm{i}\partial\varphi_{11}/\partial\tau - k\beta\partial^2\varphi_{11}/\partial\xi^2 - k^2\varphi_{11}|\varphi_{11}|^2/2(\varepsilon + 2\beta k^3) = 0. \tag{23}$$

Relation (23) is a nonlinear Schrödinger equation. A solitary wave solution of this equation is of the form (Yuen and Lake 1987)

$$\varphi_{11} = a_0 \operatorname{sech}[a_0(\Delta/\beta k)^{1/2}\xi]\exp[-\mathrm{i}a_0^2\Delta\tau] \tag{24}$$

where $\Delta = k^2/2(\varepsilon + 2\beta k^3)$. Solution (24) shows that the right-hand term in equation (16) leads to an increase of the characteristic weight and a decrease of the frequency for an envelope soliton (24).

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